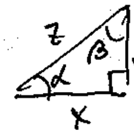
Name: MATH 3 | **Student ID:** ANSWER KEY

MATH 3: Exam 2

Problem 1. (5 points) Determine whether the following statements are **TRUE** or **FALSE**. No justification is required

(a) (1 point) If α and β are the two acute interior angles of a right triangle, then $\cos(\alpha) = \sin(\beta)$. Hint: draw a picture



$\cos(\alpha) = \frac{x}{z} = \sin(\beta)$

Answer: $\boxed{\text{TRUE}}$

(b) (1 point) The function $f(x) = \sin(\theta)$ is one-to-one.

$\sin(0) = \sin(\pi)$

Answer: $\boxed{\text{FALSE}}$

(c) (1 point) If $f(x)$ is a polynomial and $f(k) = 0$, then the remainder of $f(x)$ divided by $x - k$ is equal to 0.

"Remainder Theorem"

Answer: $\boxed{\text{TRUE}}$

(d) (1 point) An angle in standard position has infinitely many coterminal angles.

θ and $\theta + 2k\pi$, $k = \ldots, -2, -1, 0, 1, 2, \ldots$ are coterminal

Answer: $\boxed{\text{TRUE}}$

(e) (1 point) If α is the reference angle of an angle θ in standard position, then $\cos(\theta) = \cos(\alpha)$.

$\cos(3\pi/4) = -\frac{\sqrt{2}}{2}$ $\quad \cos(\pi/4) = \frac{\sqrt{2}}{2}$

Answer: $\boxed{\text{FALSE}}$

Problem 2. (8 points) Let $f(x) = x^5 + x^4 + x^3 + x^2 + x + 1$.

(a) (6 points) Divide $f(x)$ by $x^2 + x + 1$ using long division.

$$
\begin{array}{r}
x^3 \quad\quad\quad + 1 \\
x^2+x+1 \overline{\big)\, x^5 + x^4 + x^3 + x^2 + x + 1} \\
-(x^5 + x^4 + x^3) \quad\quad\quad\quad \downarrow \\
\hline
0 + x^2 + x + 1 \\
-(x^2 + x + 1) \\
\hline
0
\end{array}
$$

$$f(x) = (x^3+1)(x^2+x+1) + 0$$

(b) (2 points) Identify the quotient $q(x)$ and the remainder $r(x)$.

$q(x) = x^3 + 1 \qquad r(x) = 0$

Problem 3. (9 points) Let $f(x) = x^3 - 4x^2 + 5x - 2$.

(a) (3 points) List all possible rational zeros of $f(x)$.

$\frac{p}{q}$ where p divides -2 and q divides 1

$\frac{p}{q} = \pm 2, \pm 1$

(b) (3 points) Determine all rational zeros of $f(x)$.

Evaluate all possibilities in (a).

$$\begin{array}{r|rrrr} 2 & 1 & -4 & 5 & -2 \\ & \downarrow & 2 & -4 & 2 \\ \hline & 1 & -2 & 1 & \boxed{0} \end{array} \qquad f(2)=0$$

$$\begin{array}{r|rrrr} 1 & 1 & -4 & 5 & -2 \\ & \downarrow & 1 & -3 & 2 \\ \hline & 1 & -3 & 2 & \boxed{0} \end{array} \qquad f(1)=0$$

$$\begin{array}{r|rrrr} -2 & 1 & -4 & 5 & -2 \\ & \downarrow & -2 & 12 & -34 \\ \hline & 1 & -6 & 17 & -36 \end{array} \qquad f(-2)=-36$$

$$\begin{array}{r|rrrr} -1 & 1 & -4 & 5 & -2 \\ & \downarrow & -1 & 5 & -10 \\ \hline & 1 & -5 & 10 & -12 \end{array} \qquad f(-1)=-12$$

$x = 2$ and $x = 1$ are the only rational zeros.

(c) (3 points) Factor $f(x)$ as a product of three linear (degree 1) polynomials.

From (b), $f(2)=0$ so $(x-2)$ is a factor.

$$\Rightarrow \quad f(x) = (x-2)(x^2-2x+1)$$

$$= (x-2)(x-1)^2$$

Problem 4. (8 points) Let $f(x) = \frac{(x-1)(x-5)(x-4)}{(x-2)(x-3)(x-4)}$.

(a) (2 points) Determine the x and y intercept(s) of $f(x)$.

x-int: $(x-1)(x-5)(x-4) = 0 \iff x=1$ or $x=5$ or $x=4$

$f(4)$ is undefined. Thus, $x=1$, $x=5$ are the x-int

y-int: $f(0) = \frac{(-1)(-5)(-4)}{(-2)(-3)(-4)} = \frac{5}{6}$

(b) (2 points) Determine the vertical asymptote(s) of $f(x)$.

$x=2$ and $x=3$. Note: $x=4$ is not an asymptote because of (d)

(c) (2 points) Determine the horizontal or slant asymptote of $f(x)$.

Degree of numerator = degree of denominator $\Rightarrow$ $y=1$ is the asymptote.

(d) (2 points) Determine the removable discontinuity of $f(x)$, if one exists.

$x=4$ is removable since $(x-4)$ is a factor of both the numerator and denominator

Problem 5. (6 points) Let $f(x) = 1 - 2 \cdot 2^{-x}$.

(a) (2 points) Find the horizontal asymptote, x and y intercepts of $f(x)$.

x-int: $0 = 1 - 2 \cdot 2^{-x} \iff 2^{-x} = \frac{1}{2} \iff (\frac{1}{2})^x = (\frac{1}{2})^1 \iff \boxed{x=1}$

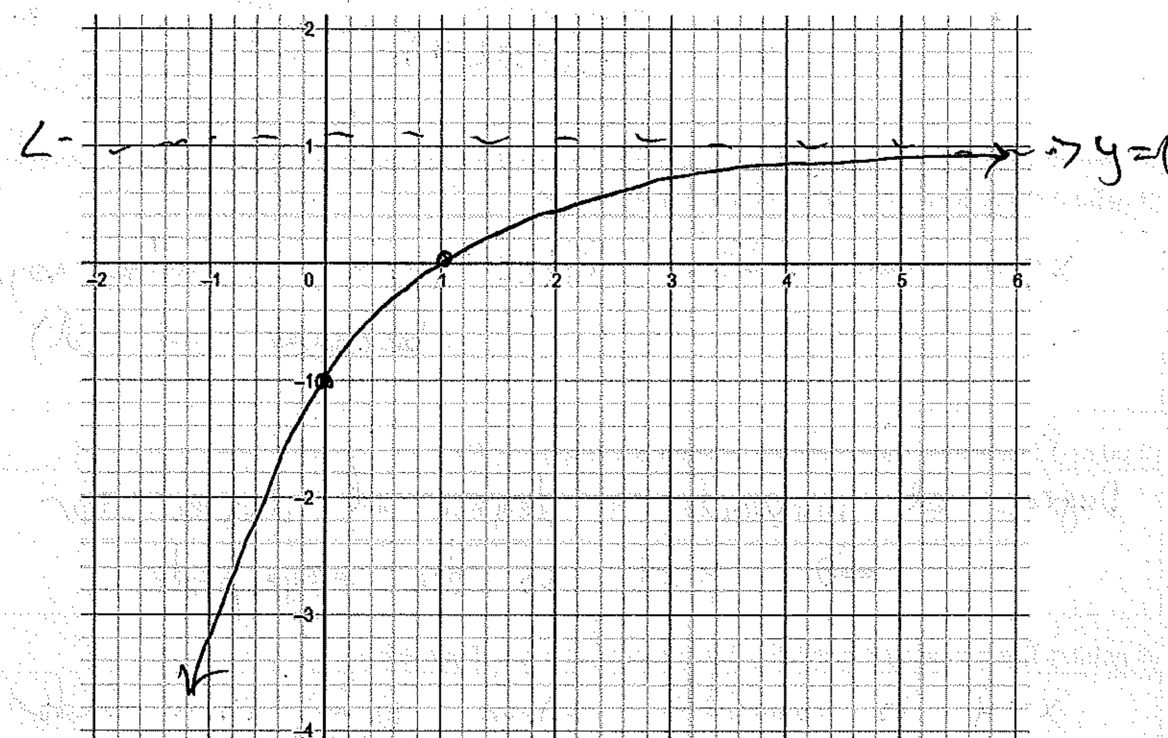
y-int: $f(0) = 1 - 2 \cdot 2^{-0} = 1-2 = -1$.

asymptote: $f(x)$ is obtained by reflecting $(\frac{1}{2})^x$ over x-axis and shifting vertically by 1. Since horiz. asymptote of $(\frac{1}{2})^x$ is $y=0$, the horiz. asymptote of $f(x)$ is $\boxed{y=1}$

(b) (2 points) Is the function $f(x) = 1 - 2 \cdot 2^{-x}$ increasing or decreasing?

Increasing

(c) (2 points) Sketch a graph of $f(x) = 1 - 2 \cdot 2^{-x}$.



Problem 6. (10 points) Find all solutions of the following equations. Be sure to check your answer for "nonsense".

(a) (5 points) Solve $e^{2x} - e^{x} - 6 = 0$.

$e^{2x} - e^{x} - 6 = 0 \iff (e^x+3)(e^x-2) = 0$

$\iff e^x + 3 = 0$ or $e^x - 2 = 0$

$\iff e^x = -3$ or $e^x = 2$

$\iff x = \ln(-3)$ or $x = \ln(2)$

$\ln(-3)$ is undefined ~~so~~

$\boxed{x = \ln 2}$

(b) (5 points) Solve $\ln(x+1) + \ln(x-1) = \ln(1)$.

$$\ln(x+1) + \ln(x-1) = \ln 1 \overset{\text{product rule}}{\Longleftrightarrow} \ln(x^2-1) = \ln 1$$

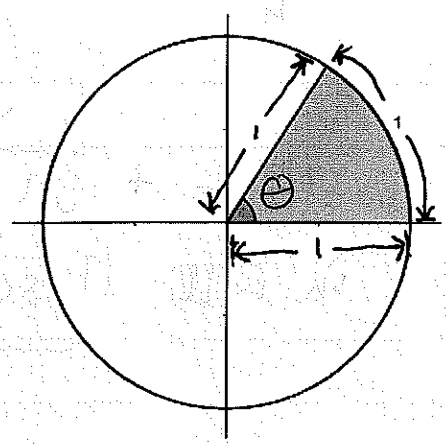
$$\overset{\text{1 to 1}}{\Longleftrightarrow} x^2 - 1 = 1$$

$$\Longleftrightarrow x^2 = 2$$

$$\Longleftrightarrow x = \pm\sqrt{2}$$

But $\ln(-\sqrt{2}-1)$ is undefined. Thus, $\boxed{x = \sqrt{2}}$

Problem 7.(4 points) Consider the following sector of the *unit circle*. The length of the arc determined by the angle is equal to one. Find the area of the sector.



The angle θ is equal to 1radian by definition:

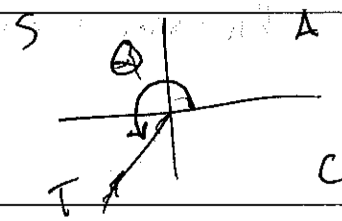
$$1 = s = r\theta = 1 \cdot \theta = \theta$$

Thus $\boxed{A = \frac{1}{2} r^2 \theta = \frac{1}{2} \cdot 1^2 \cdot 1 = \frac{1}{2}.}$

Problem 8.(12 points) Suppose that $\tan\theta = \frac{4}{3}$ and $\sin(\theta) < 0$.

(a) (2 point) Determine which quadrant θ lies in.

θ is in Q$\underline{\text{III}}$ since $\tan\theta > 0$, $\sin\theta < 0$



(b) (2 points) Find $\sec\theta$.

$\sec\theta = \pm\sqrt{\tan^2\theta + 1} = \pm\sqrt{\frac{16}{9}+1} = \pm\sqrt{\frac{25}{9}} = \pm\frac{5}{3}$

Since $\sec\theta < 0$ in Q$\underline{\text{III}}$, $\boxed{\sec\theta = -\frac{5}{3}}$

(c) (2 points) Find $\cot\theta$.

$\cot\theta = \frac{1}{\tan\theta} = \frac{3}{4}$

(d) (2 points) Find $\csc\theta$.

$\csc\theta = \pm\sqrt{1+\cot^2\theta} = \pm\sqrt{1+\frac{9}{16}} = \pm\sqrt{\frac{25}{16}} = \pm\frac{5}{4}$

Since $\csc\theta < 0$ in Q III $\boxed{\csc\theta = -\frac{5}{4}}$

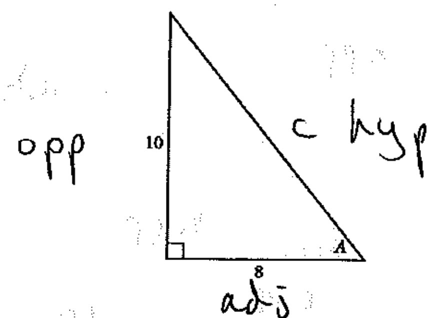
(e) (2 points) Find $\sin\theta$.

$\sin\theta = \frac{1}{\csc\theta} = -\frac{4}{5}$

(f) (2 points) Find $\cos\theta$.

$\cos\theta = \frac{1}{\sec\theta} = -\frac{3}{5}$

Problem 9. (12 points) Consider the following right triangle.



(a) (2 points) Find the length of the hypotenuse using the Pythagorean Theorem.

$8^2 + 10^2 = c^2 \iff c = \sqrt{164} = 2\sqrt{41}$

(b) (2 points) Find $\sin(A)$.

$\sin(A) = \frac{opp}{hyp} = \frac{10}{2\sqrt{41}} = \frac{5\sqrt{41}}{41}$

(c) (2 points) Find $\cos(A)$.

$\cos(A) = \frac{adj}{hyp} = \frac{8}{2\sqrt{41}} = \frac{4\sqrt{41}}{41}$

(d) (2 points) find $\tan(A)$.

$\tan(A) = \frac{opp}{adj} = \frac{10}{8} = \frac{5}{4}$

(e) (2 points) find $\cot(A)$.

$\cot(A) = \frac{4}{5}$

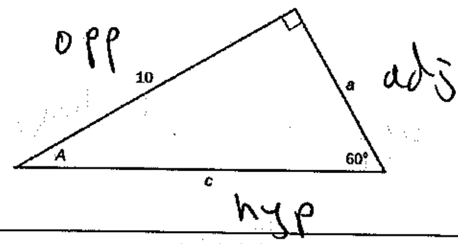
(f) (2 points)Find $\sec(A)$.

$\sec(A) = \frac{41}{4\sqrt{41}} = \frac{\sqrt{41}}{4}$

(g) (2 points) Find $\csc(A)$.

$\csc(A) = \frac{41}{5\sqrt{41}} = \frac{\sqrt{41}}{5}$

Problem 10. (6 points) Find the unknown side lengths a and c of the following triangle.



$$\frac{\sqrt{3}}{2} = \sin(60^\circ) = \frac{\text{opp}}{\text{hyp}} = \frac{10}{c} \iff \boxed{c = \frac{10 \cdot 2}{\sqrt{3}} = \frac{20\sqrt{3}}{3}}$$

$$\sqrt{3} = \frac{\sqrt{3}}{2} \Big/ \frac{1}{2} = \tan(60^\circ) = \frac{\text{opp}}{\text{adj}} = \frac{10}{a} \iff \boxed{a = \frac{10}{\sqrt{3}} = \frac{10\sqrt{3}}{3}}$$